ELSEVIER Journal of Pure and Applied Algebra 127 (1998) 119–136

JOURNAL OF
PURE AND
APPLIED ALGEBRA

The free product of groups with amalgamated subgroup malnormal in a single factor

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Communicated by C.A. Weibel; received 10 May 1994; received in revised form 22 August 1995

Abstract

We discuss groups that are free products with amalgamation where the amalgamating subgroup is of rank at least two and malnormal in at least one of the factor groups. In 1971, Karrass and Solitar showed that when the amalgamating subgroup is malnormal in both factors, the global group cannot be two-generator. When the amalgamating subgroup is malnormal in a single factor, the global group may indeed be two-generator. If so, we show that either the non-malnormal factor contains a torsion element or, if not, then there is a generating pair of one of four specific types. For each type, we establish a set of relations which must hold in the factor B and give restrictions on the rank and generators of each factor.

0. Introduction

Baumslag introduced the term *malnormal* in [1] to describe a subgroup that intersects each of its conjugates trivially. Here we discuss groups that are free products with amalgamation where the amalgamating subgroup is of rank at least two and malnormal in at least one of the factor groups. In the case that the amalgamating subgroup is free abelian of rank two, groups of this type appear as the fundamental group of certain compact 3-manifolds. In particular, a compact 3-manifold which is not the union of Seifert fibre spaces and has a separating torus in its decomposition has such a fundamental group.

Karrass and Solitar [4] showed that when the amalgamating subgroup is malnormal in both factors, the global group cannot be two-generator. When the amalgamating

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0022-4049/98/$19.00
PII S0022-4049(97)00157-6

subgroup is malnormal in a single factor, the global group may indeed be two-generator. If so, we show that either the non-malnormal factor contains a torsion element or, if not, then there is a generating pair of one of four specific types. For each type, we establish a set of relations which must hold in the factor B and give restrictions on the rank and generators of each factor. These results have several applications in low-dimensional topology, in particular to some of the problems on Kirby's famous 1978 list [5]. These applications appear in [2]. The proof of the main theorem is an expansion of techniques pioneered by Norwood [7, 8] as extended by Jones in [3]. Those papers consider cyclic amalgamating subgroups.

Section 1 contains notational and background information. Several key lemmas are provided in Section 2. Section 3 contains the statements of the main results, which are then proved in Section 4. In Section 5, we analyze the factors in light of the various types of generating pairs.

1. Background material

1.1. Definitions and notation. Here we review some of the properties of groups needed in future discussion. Let A denote a group. A subgroup S of A is called *proper* if S is non-trivial and there exist elements of A outside of S. A proper subgroup S is said to be *malnormal* in A if the intersection of subgroups aSa^{-1} and S is trivial if and only if a is an element of A outside of S. Thus, S intersects each of its non-trivial conjugates trivially. An element $a \in A$ is said to be *central* if a commutes with each element of A. A *torsion* element of A is an element a satisfying the relation $a^n = 1$ for some non-zero integer n.

1.2. The *rank* of a finitely presentable group is the minimal number of generators needed to present the group. A group A is called an *n-generator group* if A is known to have rank n. The groups of rank one are the finite cyclic groups together with the infinite cyclic group. The groups of higher rank form a much more complicated class. Indeed, determining the rank of a group is in general a difficult problem.

1.3. The following concepts are presented in greater detail in Magnus et al. [6] and are recalled here for the convenience of the reader. Let $A *_C B$ denote the free product with amalgamation of the non-trivial groups A and B along the group C, that is, C is isomorphic to a proper subgroup A_C of A and also isomorphic to a proper subgroup B_C of B. The subgroups A_C and B_C are identified with the group C via the respective isomorphisms. When the context is clear, we simply refer to the subgroup A_C of A by C, and similarly the subgroup B_C of B will be referred to by C. Note that group G contains each of A, B and C as proper subgroups. The groups A and B are called the *free factors* for the group $A *_C B$ and the group C is called the *amalgamated subgroup.*

Any element x of $A *_C B$ may be uniquely expressed in *normal form* as $x = x_1 x_2 \cdots x_m c$ where c is an element of C and the x_i are alternately elements of a fixed set T_A of right transversals for the (right) cosets of C in A and a corresponding fixed set T_B of right transversals for the (right) cosets of C in B. The coset C in either of A or B always receives the transversal represented by 1 and the transversal representative 1 never appears in the normal form word for x. If x is expressed in the above normal form, then x is said to have length m, and one writes $length(x) = m$. If x_1 is an element of A, then x is said to begin in A. If x_m is an element of A, then x is said to end in A. Similarly, if x_i is an element of B for $i = 1$ or $i = m$, then x is said to begin or end, respectively, in B. If x begins in A and ends in B, then x can be written concretely in the normal form $x = a_1 b_2 \cdots b_m c$. If x can be written in this form, then x is said to be a word of length m beginning in A and ending in B, and similarly for the other cases.

Two words x and y written in normal form are multiplied by concatenation. The product is subsequently converted to normal form by moving the elements of C to the right through the word y. If the word $x = x_1 \cdots x_n c$ ends in the same group that the word $y = y_1 \cdots y_m c'$ begins in, then the product $x_n c y_1$ is an element of one of A or B. If $x_n c y_1$ is an element of the complement of C in that factor, then xy is said to have *amalgamation* and $length(xy) = n + m - 1$. If the product $x_n c y_1 = \bar{c}$, an element of C, then the word xy has *cancellation* and the product $x_{n-1} \bar{c} y_2$ is then examined for additional cancellation and/or amalgamation and $length(xy) \leq m + n - 2$. If x ends in a different group than y begins in, there is no amalgamation or cancellation in xy and $length(xy) = m + n$.

The following result is due to Stallings [9, Theorem 4.3].

1.4. Lemma. *If n elements generate a free product with amalgamation, then there is a set of n generators in which at least one of the generators is an element of one of the free factors.*

1.5. Lemma. *Suppose that the group $A *_C B$ is generated by n elements $g_1, g_2, \ldots, g_n$. The set S_A consisting of those transversals of T_A appearing in any of the g_i together with all of the elements of C generates the group A.*

Proof. Let S denote the subgroup of A generated by S_A. Note that if a_i and a_j are in S_A then the product $a_i c a_j$ is an element of S even though the product may be rewritten in normal form as $a_\lambda \bar{c}$ where the transversal representative a_λ is not an element of S_A.

By assumption, the g_i generate $A *_C B$ so they must also generate A. An arbitrary element a of A may then be written as a word w in the g_i. As a may also be written $a = a_\lambda \tilde{c}$ for some right transversal a_λ in the set T_A and element $\tilde{c}$ of C, the uniqueness of normal form implies that the word w admits cancellations which reduce its length to 1. In particular, all occurences of elements of B cancel to produce elements of C, all of which lie in S. It follows that S actually equals A. □

1.6. Remark. A parallel argument shows that the group B may be generated by the set S_B consisting of all transversal elements of T_B that appear in at least one of the generators g_i together with the elements of C.

2. Preliminary lemmas

2.1. Remark. Let $G = A *_C B$ be a non-trivial two-generator free product with amalgamation of groups A and B along the group C, here C is not necessarily malnormal in either factor group. By Lemma 1.4 there is a pair of generators $\{g_1, g_2\}$ for G so that one of the generators is an element of group A or B. Throughout the discussion, this generator of length zero or one will be denoted g_1. By Lemma 1.5 and Remark 1.6, together with the fact that C is a proper subgroup of each of A and B, a transversal from each of $A - C$ and $B - C$ must appear somewhere in the pair $\{g_1, g_2\}$.

2.2. Remark. The following is a complete list of all possible normal forms for the pair $\{g_1, g_2\}$ described in Remark 2.1:

(i) $\{c, a_1 \cdots a_n c_0\}$,
(ii) $\{c, a_1 \cdots b_n c_0\}$,
(iii) $\{c, b_1 \cdots b_n c_0\}$,
(iv) $\{c, b_1 \cdots a_n c_0\}$,
(v) $\{ac, a_1 \cdots a_n c_0\}$,
(vi) $\{ac, a_1 \cdots b_n c_0\}$,
(vii) $\{ac, b_1 \cdots b_n c_0\}$,
(viii) $\{ac, b_1 \cdots a_n c_0\}$,
(ix) $\{bc, a_1 \cdots a_n c_0\}$,
(x) $\{bc, a_1 \cdots b_n c_0\}$,
(xi) $\{bc, b_1 \cdots b_n c_0\}$,
(xii) $\{bc, b_1 \cdots a_n c_0\}$,
(xiii) $\{ac, bc_0\}$.

2.3. Remark. Recall that given a pair of subgroups S_1 and S_2 of a group G, the group S_2 is *inner automorphic* to the group S_1 if there is an element g in G conjugating S_2 onto S_1. Given a pair of elements $\{g_1, g_2\}$ in normal form, we say that another pair is *equivalent* to the first if the two generate inner automorphic subgroups of G. Here are two operations we use to replace a generating pair by an equivalent pair which may have a different normal form. The first is to conjugate each element of the pair by an element of G and write the resulting pair in normal form. The second operation is to replace one of the elements g_1 or g_2 by its inverse, and then convert that element to normal form. These operations are all we require, although not all equivalent pairs differ by a sequence of these operations. Under this definition, equivalence forms an equivalence relation on pairs of elements. We will determine those equivalence classes of pairs capable of generating the group G.

2.4. Proposition. *A pair of elements of type* (iii)–(v), (viii), (xi) *or* (xii) *of Remark* 2.2 *is equivalent to a pair of type* (i), (ii), (vi), (vii), (ix), (x) *or* (xiii).

Proof. A pair of type (v) is equivalent to a pair of type (iii), (iv), (vii), (viii) or (xiii) by conjugating this pair by the element a_1^{-1}. If $a_1^{-1}aca_1$ is an element of C then an equivalent pair of type (iii) or (iv) results. If $a_1^{-1}aca_1$ is an element of the set $A-C$ then an equivalent pair of type (vii), (viii) or (xiii) results.

A parallel argument shows that a pair of type (xi) is equivalent to a pair of type (i), (ii), (ix), (x) or (xiii) by conjugating the pair by the element b_1^{-1}.

Conjugating a pair of type (iii) by the element b_1^{-1} shows this pair is equivalent to a pair of type (i), (ii), (ix), (x) or (xiii).

Replacing the element g_2 with the inverse g_2^{-1} shows a pair of type (iv) to be equivalent to one of type (ii), a pair of type (viii) to be equivalent to one of type (vi) and a pair of type (xii) to be equivalent to a pair of type (x). □

2.5. Claim. *A generating pair of type* (i) *in Remark* 2.2 *is equivalent to a pair of type* (ii), (vi), (vii), (ix), (x) *or* (xiii).

Proof. Conjugating the pair of type (i) by the element a_1^{-1} yields a pair of elements of type (iii), (iv), (vii), (viii) or (xiii). A type (iv) pair is equivalent to a pair of type (ii), and a type (viii) pair is equivalent to a pair of type (vi) by proof of Proposition 2.4. Notice that if a type (iii) pair results then the length of the element g_2 has been reduced by two. Conjugating this resultant type (iii) pair (after first converting the pair to normal form) by the element b_1^{-1} yields an equivalent pair of type (i), (ii), (ix), (x) or (xiii) by proof of Proposition 2.4. Note that if a type (i) pair results then the length of g_2 in this new pair is four less than the length of g_2 in the original pair of type (i). In this way, a type (i) pair is equivalent to a pair as desired in Claim 2.5, or has been conjugated so that the length of g_2 has been reduced to one or three. If the length of g_2 is one, the result is not a generating pair as there is not a transversal representative from each of T_A and T_B. Thus the length of g_2 is three. Now, conjugation by the element x_1^{-1} will yield a pair of elements of type (iv), (vii), (viii) or (xiii). A pair of type (ix) may be converted to an equivalent one of type (ii) and a pair of type (viii) may be converted to an equivalent pair of type (vi) as in proof of Proposition 2.4. □

The following sequence of propositions will be used to establish Theorem 3.1.

2.6. Proposition. *Let A be a torsion-free group containing a proper malnormal subgroup C. Then A has no non-trivial central elements and the relation $a^p \in C$ holds if and only if the element a of A is an element of C.*

Proof. Assume that a is a central element of A. Then the relation $ac=ca$ holds for every element c in C, and so aCa^{-1} coincides with C. Thus, a must be an element of C. But now gCg^{-1} contains the element a for all elements g of A, implying that $a=1$.

If an element a of $A-C$ satisfies the relation $a^p=\bar{c}$ where $\bar{c}$ is a non-trivial element of C, then elements a and $\bar{c}$ commute:

$$a\bar{c}=aa^p=a^pa=\bar{c}a.$$

Thus, the subgroup aCa^{-1} contains the non-trivial element $\bar{c}$, again contradicting the malnormality of C in A. □

2.7. Proposition. *Let C be a subgroup of the group B. If the element b in $B-C$ has the property that b^n is an element of C, then there is a unique minimal positive integer m such that $n=im$ for some integer i.*

Proof. Assume that m is the minimal positive integer for which b^m is an element of C. Assume that b^n is also an element of C for some $n \geq m$. Then the relation $n=mi+r$ holds for integers i and r satisfying $0 \leq r \leq m-1$. The following relation also holds:

$$b^n=b^{mi+r}=b^{mi}b^r.$$

Since b^n and b^{im} are elements of C, then b^r must also be an element of C, contrary to the choice of m unless $r=0$. □

2.8. Proposition. *Let C be a malnormal subgroup of group A. Choose and fix a set of transversal elements T_A for the right cosets of C in A. Let a_i and a_j be two arbitrarily chosen non-trivial elements of T_A. If there exists an element c of C for which the word a_ica_j is an element of C, then this element in unique.*

Proof. Let c be an element of C satisfying $a_ica_j=\bar{c}$ where $\bar{c}$ is an element of C. Assume for sake of contradiction that there is another element c' for which $a_ic'a_j$ is an element of C. The element a_i may be written $a_i=\bar{c}a_j^{-1}c^{-1}$ using the element c. Substituting this into $a_ic'a_j$ yields the expression $\bar{c}a_j^{-1}c^{-1}c'a_j$. This expression is an element of C if and only if the expression $a_j^{-1}c^{-1}c'a_j$ is an element of C. By the malnormality of C the intersection of $a_j^{-1}Ca_j$ and C can only contain the identity, thus $c'=c$. □

2.9. Proposition. *Let $G=A*_CB$ be the free product of the torsion-free group A and a group B, amalgamated over a group C that is malnormal in A. Let g be an element of G satisfying length(g)$=n \geq 3$, then either the element g^p satisfies length(g^p)$\geq n-2$ and the elements g^p and g begin and end in the same group or the group B contains torsion elements and whenever g^p has length less than $n-2$ it has length zero.*

Proof. Assume that *length*(g^p) $\leq n-1$ for some non-zero integer p. In uncancelled form the length of g^p is $|p|n$. If the element g begins and ends in different groups, then g^p experiences no cancellation or amalgamation. Thus, g begins and ends in the same group and so n is, in fact, odd, say $length(g) = 2j + 1$. Note for latter use that the element in the middle of g has index $j+1$. We first examine the situation in which g begins and ends in group B. The word g^p when written in uncancelled form looks like

$$g^p = b_1 \cdots b_n c_0 b_1 \ldots \cdots b_n c_0 b_1 \cdots b_n c_0.$$

If $b_n c_0 b_1$ is an element of $B - C$, then $length(g^p) = n|p| - (|p| - 1)$ which is at least n, since n is at least 3 and $|p| \geq 2$. Thus, it must be that $b_n c_0 b_1$ is an element c_1 of C. Now g^p looks like

$$g^p = b_1 \cdots a_{n-1} c_1 a_2 \cdots a_{n-1} c_1 a_2 \cdots b_n c_0.$$

If $a_{n-1} c_1 a_2$ is an element of $A - C$, then $length(g^p) = n|p| - 3(|p| - 1)$. Again, since $n \geq 3$, this word has length at least n. Thus $a_{n-1} c_1 a_2$ is an element c_2 of C.

Continue inductively. If for some integer I satisfying $0 \leq I \leq j - 1$, the word $x_{n-i} c_i x_{i+1}$ is an element c_{i+1} of C for all $i < I$ while the element $x_{n-I} c_I x_{I+1}$ is an element of $A - C$ or $B - C$, then $length(g^p) = n|p| - (2i + 1)(|p| - 1)$, which for this range of i values is at least n. Thus, for g^p to have length less than n, the phrase $x_{n-i} c_i x_{i+1}$ must be an element c_{i+1} of C for all i satisfying $0 \leq i \leq j - 1$. The word g^p cancels to the following form:

$$g^p = b_1 \cdots x_j (x_{j+1} c_j)^p c_j^{-1} x_{j+2} \cdots b_n c_0.$$

This expression experiences cancellation if and only if $(x_{j+1} c_j)^p$ is an element of C. By Proposition 2.6 this only occurs if the element x_{j+1} appearing in the middle of g is an element of B. If the element $(x_{j+1} c_j)^p c_j^{-1}$ is the element c_{j+1} of C, then the relations $a_{j+2} c_{j-1} a_j = c_j$ and $a_j c_{j+1} a_{j+2} = c_{j+2}$ imply that the following relation holds:

$$a_{j+2} c_{j-1} c_{j+2} a_{j+2}^{-1} c_{j+1}^{-1} = c_j.$$

By the malnormality of C in A, this occurs only if the relation $c_{j-1} = c_{j+2}^{-1}$ holds, and so it follows that $c_j = c_{j+1}^{-1}$. Finally, this yields the relation

$$(b_{j+1} c_j)^p c_j^{-1} = c_{j+1} = c_j^{-1}.$$

This relation implies that the element $b_{j+1} c_j$ is a torsion element of B. Thus, if B is a torsion free group then the length of g^p is at least $n-2$. The group B may contain torsion elements, in which case we note that the relation $x_{n-i} c_i x_{i+1} = c_{i+1}$ may be rewritten as $xi + 1ci + 1^{-1} x_{n-i} = c_i^{-1}$. Since we have $c_{j+1} = c_j^{-1}$ this means that $a_j c_j^{-1} a_{j+2} = c_{j-1}^{-1}$. Now the relations $xi + 1ci + 1^{-1} x_{n-i} = c_i^{-1}$ force the element g^p to cancel all the way to the element c_0^{-1}.

Essentially, the same argument holds in the case that g begins and ends in the group A. □

2.10. Remark. If G is a group as in Proposition 2.9 in which the factor B is torsion free, then any element x in G satisfying the relation $x^p \in C$ is an element of the group B.

3. The main result and a corollary

The following theorem is the main result of this paper.

3.1. Theorem. *Let $G = A *_C B$ be the free product with amalgamation of the torsion-free group A and the group B. If the amalgamated subgroup C is of rank at least two and malnormal in A and if G has rank two, then either B contains torsion elements or G has a generating pair of type* (ii), (ix), (x), *or* (xiii) *of Remark* 2.2. *in which each element has length at most three.*

As a corollary we obtain a result which is essentially of [4, Theorem 6].

3.2. Corollary. *Let G be the free product with amalgamation of the torsion-free groups A and B amalgamated over a group which is malnormal in each factor. Then the group G may not be generated by any two of its elements.*

Proof. If the rank of C is at least two, then the result follows from Theorem 3.1 and Proposition 2.6. Assume the rank of C is one. By [3, Lemma 2.14], condition (iii) of [3, Lemma 2.8] may be replaced by the condition that x_i and u commute if and only if the product $x_i u^p x_i^{-1}$ is an element of C for some non-zero integer p without affecting its validity. Here x_i denotes a transversal of T_A or T_B and u is a generator for C. If C is malnormal in A and B then the product $x_i u^p x_i^{-1}$ is in C if and only if $p = 0$. By Proposition 2.6, the center of A and B is trivial. By 2.18, if the element $g \in G$ satisfies the relation $g^p \in C$, then g is an element of A, B or C. By Proposition 2.6, this relation cannot occur in either group A or B unless the element g is a torsion element in that group. Since these groups are torsion free, the condition $g^p \in C$ implies $g^p = 1$ holds vacuously for the group G. The result now follows from [3, Lemma 2.4]. □

4. A proof of the main results

Throughout this section, let G denote the free product of groups A and B amalgamated along C, a group of rank at least two. The group A is torsion free, and group C is malnormal in A. By Claim 2.5 we need only examine the subgroup of G generated by a pair of elements of type (ii), (vi), (vii), (ix), (x) and (xiii) as in Remark 2.2.

We begin by establishing notation.

4.1. Remark. Let g_1 and g_2 be two elements of group G. In all references to these elements it is to be implicit that the pair $\{g_1, g_2\}$ appears in Remark 2.2. These two elements generate a subgroup of G. Any element g of this subgroup, if not a power of one of g_1 or g_2 may be written as a word w in the generators g_1 and g_2 with the following form:

$$w = g_1^{p_1} g_2^{p_2} \cdots g_2^{p_{2k}} g_1^{p_{2k+1}}.$$

Here k denotes a positive integer and each integer p_i for which i satisfies the inequality $2 \leq i \leq 2k$ is non-zero. We allow the possibility that the integers p_1 and p_{2k+1} are zero. Although the word w is not necessarily a unique representative for the element g it can often be used to evaluate the length of the element in G represented by w, which is uniquely determined.

Given a fixed word w as above, we now associate a series of integers to w. Let P denote the associated sequence $\{p_2, p_4, \ldots, p_{2k}\}$. The integer σ denotes the number of times that the sequence P changes algebraic sign. For example, if w is the word $g_1^8 g_2^4 g_1^{-2} g_2^{227} g_1 g_2^{-88} g_1 g_2$, then the sequence P is $\{4, 227, -88, 1\}$ and the associated integer σ is 2. Note that the relation $0 \leq \sigma \leq k-1$ holds. Further, for each integer $2i+1$ satisfying $0 \leq i \leq k$, let ε_{2i+1} denote the length of the phrase $g_1^{p_{2i+1}}$.

The word w may not be in normal form. In the given form, the word w has an *uncancelled length* computed by counting the number of transversals appearing in the word w. Denote the uncancelled length of the word w by the letter λ and note that λ is an upper bound for the length of the element that w represents in the group G. In particular, λ denotes the following sum:

$$\lambda = \left(\sum_{i=1}^{i=k} |p_{2i}| \right) n + \sum_{i=0}^{i=k} \varepsilon_{2i+1}.$$

Here n denotes *length*(g_2). Note that *length*(w) can only mean the length of the element in G represented by w when that element is written in normal form.

4.2. Lemma. *Let G be as in Theorem* 3.1. *If n is greater than two, then a pair of elements of type* (ii) *of Remark* 2.2 *cannot generate G.*

Proof. Assume that the pair $\{g_1, g_2\}$ of type (ii) will generate G. Since g_2 begins and ends in different groups, no power of g_2 experiences either amalgamation or cancellation. For this same reason, any phrase of form $g_2^p g_1^q g_2^r$ has cancellation or amalgamation if and only if the integers p and r have opposite algebraic sign.

In uncancelled form the phrase $g_2^{-1} g_1^q g_2$ looks like

$$c_0^{-1} b_n^{-1} \cdots a_1^{-1} c^q a_1 \cdots b_n c_0.$$

Since the subgroup C is malnormal in A, amalgamation occurs in the phrase $a_1^{-1} c^q a_1$ but no cancellation occurs. Thus such a phrase has length $2n-1$.

In uncancelled form the phrase $g_2 g_1^q g_2^{-1}$ looks like

$$a_1 \cdots b_n c_0 c^q c_0^{-1} b_n^{-1} \cdots a_1^{-1}.$$

The element $b_n c_0 c^q c_0^{-1} b_n^{-1}$ may be an element of either $B - C$ or C. If the former occurs, then this phrase experiences amalgamation only and has length $2n-1$. If the latter occurs, then the phrase $b_n c_0 c^q c_0^{-1} b_n^{-1}$ is an element c' of C and the phrase $a_{n-1} c' a_{n-1}^{-1}$ experiences amalgamation but not cancellation by the malnormality of C in A. Thus, a phrase of form $g_2 g_1^q g_2^{-1}$ has a minimum length of $2n-3$.

As the group C has rank at least two, the element g_1 does not generate all of C. Thus, some elements of length zero must arise only as a word w in the form of Remark 4.1.

There are now three cases to consider depending on the integer σ associated to the word w.

Case 1: The integer σ is even, say $\sigma = 2s$. Here, after cancellation, the length of w satisfies the following series of inequalities:

$$length(w) = \lambda - 4s \geq kn - 2(k-1)/geqk(n-2) + 2.$$

Since the length of g_2 is at least four, the word w has length at least four.

Case 2: The integer σ is odd, say $\sigma = 2s+1$ and there are more occurrences of phrase $g_2^{-1}g_1^q g_2$ than of $g_2 g_1^q g_2^{-1}$. As $\sigma \leq k-1$ then $2s \leq k-2$, and so the length of w satisfies:

$$length(w) = \lambda - 4s - 1 \geq kn - 2(k-2) - 1 = k(n-2) + 3 \geq 3.$$

Thus, all such words have length at least five.

Case 3: The integer σ is odd, say $\sigma = 2s+1$ and there are more occurrences of phrase $g_2 g_1^q g_2^{-1}$ than of $g_2^{-1} g_1^q g_2$. The length of w satisfies the following relation:

$$length(w) = \lambda - 4s - 3 \geq kn - 2(k-2) - 3 \geq k(n-2) + 1.$$

Such a word has a length of at least three.

Thus, no words of length less than three arise as a word w of the form of Remark 4.1, hence the pair $\{g_1, g_2\}$ cannot generate G. □

4.3. Lemma. *Let* G *be as in Theorem* 3.1. *A pair of elements of type* (vi) *in Remark* 2.2 *cannot generate the group* G.

Proof. Assume that the pair $\{g_1, g_2\}$ is a pair of elements of type (vi) of Remark 2.2 and generate the group G. By Proposition 2.6, any non-trivial power of generator g_1 is an element of $A - C$. As g_2 begins and ends in different groups, the length of g_2 is at least 2 and no non-trivial power of g_2 experiences either amalgamation or cancellation. Thus, all elements of C must arise as a word w in the form of Remark 4.1.

We may assume that the element $(ac)^p a_1$ is in $A - C$ for all integers p, for if $(ac)^p a_1$ is an element of C then the pair $\{g_1, g_1^p g_2\}$ is a pair with normal form of type (vii) in Remark 2.2 that generates the same subgroup of G that $\{g_1, g_2\}$ does. We now assume that phrases of the form $g_2 g_1^p g_2$ and $g_2^{-1} g_1^p g_2^{-1}$ have amalgamation only.

The phrase $g_2 g_1^p g_2^{-1}$ has neither cancellation nor amalgamation since g_2 ends in B and g_1^p is an element of $A - C$.

The phrase $g_2^{-1} g_1^p g_2$ may have cancellation as well as amalgamation depending on whether the element $a_1^{-1}(ac)^p a_1$ is an element of C or not.

4.3.1. Claim. *If the relation* $a_1^{-1}(ac)^p a_1 \in C$ *holds for any integer* p, *then it holds for* $p = 1$.

Proof. Assume that $a_1^{-1}(ac)a_1$ is not an element of C. Then by Proposition 2.6, the element $(a_1^{-1}(ac)a_1)^p = a_1^{-1}(ac)^p a_1$ is also not an element of C for all non-zero integers p. □

Thus, if the relation $a_1^{-1}(ac)^p a_1 \in C$ occurs we conjugate our type (vi) generating pair to obtain an equivalent type (ii) generating pair. We now assume that this relation occurs for no non-zero integer p.

4.3.2. If the relation $a_1^{-1}(ac)^p a_1 \in A - C$ holds for all non-zero integers p, then the phrase $g_2^{-1} g_1^p g_2$ has only amalgamation and we may calculate the length of a word w in the form of Remark 4.1. There are three subcases, depending on the parity of σ:

Case 1a: The integer σ is even, say $\sigma = 2s$. Then the length of w satisfies:

$$length(w) = \lambda - \sigma - (k - 1 - \sigma) \geq kn + 1.$$

Since $n \geq 2$ and k is at least one, such a word has length at least three.

Case 1b: The integer σ is odd, say $\sigma = 2s + 1$ and there are more occurrences of the phrase $g_2 g_1^p g_2^{-1}$ than of the phrase $g_2^{-1} g_1^p g_2$.

Then $length(w)$ satisfies the relation:

$$length(w) = \lambda - 2s - (k - 1 - \sigma) \geq kn + k + (k - 1) - (k - 1) + 1.$$

Since $n \geq 2$ and k is at least one, such a word has length at least four.

Case 1c. The integer σ is odd, say $\sigma = 2s + 1$ and there are more occurrences of the phrase $g_2^{-1} g_1^p g_2$ than of the phrase $g_2 g_1^p g_2^{-1}$.

Then $length(w)$ satisfies the relation:

$$length(w) = \lambda - 2s - 2 - (k - 1 - \sigma) \geq kn.$$

Since $n \geq 2$ and k is at least one, such a word has length at least two.

Thus, if $a_1^{-1}(ac)^p a_1$ is an element of $A - C$ for each non-zero integer, then g_1 and g_2 do not generate elements of C, and hence fail to generate G.

If $a_1^{-1}(ac)a_1$ is an element of C, then this type (vi) pair is equivalent to a pair of type (ii) in Remark 2.2. □

4.4. Lemma. *Let G be as in Theorem* 3.1, *then G cannot be generated by a pair of elements of type* (vii) *of Remark* 2.2.

Proof. The element g_1^p is an element $A - C$ by Proposition 2.6 and so has length one. The element g_2 begins and ends in group B and so by Remark 2.10 the element g_2^p has length at least $n - 2$, which is at least one for all integers p. So again all elements of C must arise as a word w in the form of Remark 4.1.

As each power $g_2^{p_{2i}}$ begins and ends in B, while each power $g_1^{p_{2i+1}}$ is an element of $A - C$ the word w above experiences no amalgamation or cancellation outside of that in the individual phrases $g_1^{p_{2i+1}}$ and $g_2^{p_{2i}}$. Thus, the length of a word w in the form of Remark 4.1 is at least k and so elements of length zero do not arise from this generating pair. □

4.5. Lemma. *Let G be as in Theorem* 3.1. *If the group B contains no torsion elements and the integer n is greater than three, then a pair of elements of type* (ix) *in Remark* 2.2 *cannot generate the group G.*

Proof. Assume for the sake of contradiction that the pair $\{g_1, g_2\}$ of type (ix) generate the group G and that the group B contains no torsion elements.

By Remark 2.10 the element g_2^p begins and ends in group A and has length at least $n-2$.

4.5.1. Claim. *If g_1^p is not an element of C for all non-zero p, then no type* (ix) *pair of generators generate G, even if $n=3$.*

Proof. Assume that g_1^p is not an element of C for all nonzero p and that the pair g_1 and g_2 generate G. Then an element of C appears as a word w in the form of Remark 4.1. As $g_1^{p_i}$ begins and ends in a different group than that which $g_2^{p_{2i+1}}$ begins and ends in, no word w experiences either amalgamation or cancellation outside of cancellation in the individual phrases $g_1^{p_i}$ and $g_2^{p_{2i+1}}$. Thus, the length of w is at least k, contrary to the assumption that g_1 and g_2 generate C. □

4.5.2. Claim. *The following conditions are mutually exclusive*:

(A) *g_2^p has length less than $pn-p+1$,*

(B) *$g_2 g_1^l g_2$ has length less than $2n-1$.*

Proof. This follows from Proposition 2.8 as follows: If condition (A) occurs then the element c_0 is the unique element such that $a_n c_0 a_1$ is an element of C. If condition (B) occurs, then $c_0 g_1^l$ must be the unique element of C for which $a_n c_0 g_1^l a_1$ is an element of C. As the group B contains no torsion elements the elements c_0 and $c_0 g_1^l$ are different and so conditions (A) and (B) cannot simultaneously occur. This proves Claim 4.5.2. □

4.5.3. Claim. *If neither* (A) *nor* (B) *or Claim* 4.5.2. *hold then the pair $\{g_1, g_2\}$ cannot generate G.*

Proof. The element g_2^p has length $pn-p+1$ and the phrase $g_2 g_1^l g_2$ has length $2n-1$.

We compute the length of a word w from Remark 4.1 to be

$$\mathit{length}(w) \geq \lambda - \left(k - 1 - \sum_{i=1}^{i=k-1} \varepsilon_{2i+1}\right)$$

$$\geq nk - k + 1 + \varepsilon_1 + \varepsilon_{2k+1} + 2\sum_{i=1}^{i=k-1} \varepsilon_{2i+1} \geq 1.$$

Thus, all elements of length zero are obtained as a power of g_1. By Proposition 2.7 there exists an integer m such that all elements of C arise as a power of g_1^m, contrary to the rank of C being at least two.

4.5.4. Remark. We may assume that condition (A) of Claim 4.5.2 always occurs. If condition (B) occurs, then the type (ix) pair generates the same group that the pair $\{g_1, g_2 g_1^l\}$ generates. This latter pair is also of type (ix) and satisfies condition (A).

4.5.5. Claim. *If $n \geq 5$, then a type (ix) pair cannot generate G.*

Proof. By Proof of Claim 4.5.3 there is a minimal positive integer m so that $g_1^m \in C$. We will show that g_1 and g_2 cannot generate the entire subgroup C of G. It has already been established that no power of g_2 is a non-trivial element of C. By Proposition 2.9 such power has a length of at least three.

By Claim 4.5.2, phrases of form $g_2 g_1^{p_i} g_2$ have no cancellation and have amalgamation only if $g_1^{p_i}$ is an element of C. Thus, *length*(w) satisfies

$$length(w) \geq \lambda - \left(k - 1 - \sum_{i=1}^{i=k-1} \varepsilon_{2i+1}\right)$$

$$\geq k + \varepsilon_1 + \varepsilon_{2k+1} + 2\sum_{i=1}^{i=k-1} \varepsilon_{2i+1} - k + 1 \geq 1.$$

Thus, all elements of length zero are obtained as powers of g_1. Now by Proposition 2.7, there exists an integer m such that all elements of C arise as a power of g_1^m, contrary to the rank of C being at least two. $\square$

4.6. Lemma. *Let G be as in Theorem* 3.1. *If $n \geq 6$, then a pair of elements of type* (x) *in Remark* 2.2 *cannot generate G. If $n = 4$ the pair may generate if the phrase $g_2 g_1^p$ experiences cancellation for some power p.*

Proof. Assume that G may be generated by a pair of elements g_1 and g_2 of type (x). As g_2 begins and ends in different groups the word g_2^p has no amalgamation or cancellation for any non-zero integer p.

The phrase $g_2^{-1} g_1^p g_2$ may have amalgamation if the element g_1^p is an element of C, but has no cancellation by the malnormality of group C in A. If the element g_1^p is an element of the set $B - C$, then this phrase has no cancellation or amalgamation. In either case, the minimum length of the phrase $g_2^{-1} g_1^p g_2$ is then $2n - 1$. Notice that maximum length reduction occurs if g_1^p is an element of C.

The phrase $g_2 g_1^p g_2^{-1}$ may have cancellation as well as amalgamation. If the word $b_n c_0 g_2^m c_0^{-1} b_n^{-1}$ is an element of $c_1 \in C$ the phrase $a_{n-1} c_1 a_{n-1}^{-1}$ experiences amalgamation. No further cancellation may occur by the malnormality of C in A and so the phrase $g_2 g_1^p g_2^{-1}$ has a minimum length of $2n - 3$.

We may assume that the word $g_2 g_1^p$ has no cancellation for any integer p. If this word has cancellation, then the pair of elements $\{g_1, g_2 g_1^p\}$ is of type (ix) in Remark 2.2 and generates the same subgroup of G. By Lemma 4.5, such a pair cannot generate G unless the length of $g_2 g_1^p$ is exactly three. We assume phrases of the form $g_2 g_1^p g_2$ and $g_2^{-1} g_1^p g_2^{-1}$ have amalgamation if g_1^p is not an element of C and have no amalgamation or cancellation otherwise. Thus, the length of such a phrase is always $2n$.

We now evaluate the minimum length of a word w with the form of Remark 4.1. As usual, there are three cases depending on the parity of σ.

Case 1: The integer σ is the even integer $2s$. The length of w satisfies:

$$length(w) \geq \lambda - 4s \geq kn - 2(k - 1).$$

Since n is at least two, all such words have length at least two.

Case 2: The integer σ is the odd integer $2s + 1$ and w has one more occurrence of $g_1^{-1} g_2^{p_i} g_1$ than of $g_1 g_2^{p_i} g_1^{-1}$.

Here, the length of w satisfies:

$$length(w) = \lambda - 4s - 1 \geq kn - 2(k - 2) - 1.$$

The last expression simplifies to $k(n-2)+3$. Since $n \geq 2$, such a word has a minimal length of three.

Case 3: The integer σ is odd and w has one more occurrence of $g_1 g_2^{p_i} g_1^{-1}$ than of $g_1^{-1} g_2^{p_i} g_1$.

Here, the length of w satisfies:

$$length(w) = \lambda - 4s - 3 \geq kn - 2(k - 2) - 3.$$

This last expression simplifies to $k(n - 2) + 1$, so the minimum length of such a word is one.

Thus, the group C must be generated by powers of g_1. Let m be the minimal positive integer for which g_1^m is an element of C. By Proposition 2.7, all powers of g_1 in C have form g_1^{mi} for some integer i. Thus, C is generated by g_1^m, contrary to the rank of C being at least two. □

Proof of 3.1. Let G be as in Theorem 3.1 and assume that the group B is torsion free and contains no central elements. Then Propositon 2.4, Claim 2.5, Lemmas 4.2–4.6 show that G has a generating pair of type (ii), (ix), (x) or (xiii) of Remark 2.2. in which each element has length at most three. □

5. The factors

In this section, G denotes the two-generator free product of torsion-free groups A and B amalgamated over the group C. We assume the group C has rank at least two and is a malnormal subroup of A. By Theorem 3.1 the group G may be presented using a pair of generators having one of four special types. For each type, we establish a set of relations which must hold in the factor B and give restrictions on the rank and generators of each factor.

By Theorem 3.1, any pair of generators for G is equivalent to one of four types. We list them here with a slight change of notation:

$$p_1\colon \{bc, ac_0\},$$

$$p_2\colon \{c, a_1b_2c_0\},$$

$$p_3\colon \{bc, a_1b_2c_0\},$$

$$p_4\colon \{bc, a_1b_2a_3c_0\}.$$

5.1. The generating pair of type p_1

A generator g for one of the factor groups A or B is said to be *peripheral* if it lies in the amalgamating subgroup C. It follows from Lemma 1.5. that in this case, the group A has a presentation in which all but one of the generators are peripheral. By Remark 1.6, the group B satisfies the same condition.

5.1.1. Proposition. *If G has a generating pair of type p_1 and C is abelian, then there is an element of C which is central in B.*

Proof. Let g_1 denote generator ac_0 and g_2 denote generator bc. By Proposition 2.6, the element g_1^p is an element of C for no non-zero integer p. Therefore, a word w in the form of Remark 4.1 will never experience cancellation or amalgamation unless the element g_2^p is an element of C for some non-zero integer p. Let m denote the smallest positive integer for which this occurs, and denote g_2^m by $\tilde{c}$. Note that m is greater than one, since g_2 is not an element of C and is non-trivial as B has no torsion. The elements g_2 and $\tilde{c}$ commute as seen below:

$$g_2\tilde{c} = g_2g_2^m = g_2^mg_2 = \tilde{c}g_2.$$

By Remark 1.6, the group B is generated by b and the elements of C. All elements of C commute with $\tilde{c}$ since C is abelian. The relation $g_2\tilde{c} = \tilde{c}g_2$ yields

$$bc\tilde{c} = \tilde{c}bc.$$

Since C is abelian we then have

$$b\tilde{c}c = \tilde{c}bc.$$

The last relation implies that b and $\tilde{c}$ commute. Now $\tilde{c}$ commutes with the generators of B and hence is central in B. $\square$

5.2. The generating pair of type p_2

Here, by Lemma 1.5. and Remark 1.7, the factor groups A and B have a presentation in which all but one of the generators are peripheral.

5.2.1. Proposition. *There exists a minimal integer m such that $b_2c^mb_2^{-1}=c'$ is an element of C.*

5.2.2. Corollary. *If G may be presented using a pair of type p_2 then B is a quotient of the group with the following presentation:*

$$\langle b_2, C | b_2c^mb_2^{-1}=c' \rangle.$$

Proof of 5.2.1. We assume there exists no such integer m. We compute the length of words in the form of Remark 4.1 generated by the elements $g_1=c$ and $g_2=a_1b_2c_0$. We note that $a_1^{-1}c^ia_1$ is an element of $A-C$ for all integers i. By hypothesis, the element $b_2c^ib_2^{-1}$ is an element of $B-C$ for all non-zero integers i. Thus, the length of a word w in the form of Remark 4.1 is

$$\mathit{length}(w)=2\left(\sum_{i=1}^{i=k}|p_{2i}|\right)-\sigma\geq 2k-(k-1)\geq 2.$$

Thus, such words never have length one, a contradiction. □

5.3. The generating pair of type p_3

Now it follows from Lemma 1.5. that the group A has a presentation in which all but one of the generators are peripheral. Now, by Remark 1.7, the group B has a presentation in which all but two of the generators are peripheral.

By Claim 4.6.1 there exists a positive minimal integer m for which the element $(bc)^m$ is an element $\tilde{c}$ of C. As in Proposition 5.2.1, There exists a minimal integer n such that $b_2\tilde{c}^nb_2^{-1}=c'$, an element of C.

5.3.1. Proposition. *There exists a positive minimal integer u such that the element $(b_2c_0bcc_0^{-1}b_2^{-1})^u=\bar{c}$ is an element of C.*

5.3.2. Corollary. *If G may be presented using a pair of type p_2 then B is a quotient of the group with the following presentation:*

$$\langle b, b_2, C | (bc)^m=\tilde{c}, b_2\tilde{c}^nb_2^{-1}=c', (b_2c_0bcc_0^{-1}b_2^{-1})^u=\bar{c} \rangle.$$

Proof of 5.3.1. We assume the integer u does not exist and compute the length of a word w in Remark 4.1. Note that the length of a word is minimized if each occurrence of $(bc)^{p_{2i+1}}$ is an element of C:

$$\mathit{length}(w)=2\left(\sum_{i=1}^{i=k}|p_{2i}|\right)-\sigma\geq 2k-(k-1)\geq 2.$$

Thus, the only elements of G which have length one are powers of the element bc, a contradiction. □

5.4. The generating pair of type p_4

Again, by Claim 4.9.1 there is a minimal positive integer m for which $(bc)^m$ is an element $\tilde{c}$ of C.

5.4.1. Proposition. *If a pair of type p_4 generates the group G, then $a_3c_0a_1 = c_1$, an element of C. Moreover, there exists a minimal positive integer v so that $(b_2c_1)^v = \bar{c}$, an element of C.*

5.4.2. Corollary. *If G may be presented using a pair of type p_4 then B is a quotient of the group with the following presentation:*

$$\langle b, b_2, C \mid (bc)^m = \tilde{c}, (b_2c_1)^v = \bar{c} \rangle.$$

5.4.3. Corollary. *The generating pair p_4 may be expressed as the generating pair $\{bc, a_1b_2'a_1^{-1}c_0^{-1}\}$, where $b_2' = b_2c_1$.*

Proof. By Proposition 5.4.1, solving the equation $a_3c_0a_1 = c_1$ for the element a_3 establishes the corollary. □

Proof of 5.4.1. If the element $a_3c_0a_1$ is in $A - C$, then the length of the element g_2^p is $2p + 1$ for all non-zero integers p. We compute the length of a word in the form of Remark 4.1:

$$\mathit{length}(w) \geq \lambda - \sigma \geq 2k + k - (k - 1) = 2k + 1.$$

Such a word cannot have length zero. In order to generate all of the group C, the element $a_3c_0a_1$ must be an element of C.

We now assume that there is no integer v for which $(b_2c_1)^v$ is an element of C. We compute the length of a word w in the form of Remark 4.1:

$$\mathit{length}(w) \geq \sum_{i=1}^{i=k} \varepsilon_{2i+1} + 3k - \sigma \geq 3k - (k - 1).$$

Such a word has a length of at least one. Thus, in order to generate the entire group C, the relation $(b_2c_1)^v \in C$ must occur for some integer v. □

Now it follows from Lemma 1.5. that the group A has a presentation in which all but one of the generators are peripheral and by Remark 1.7, the group B has a presentation in which all but two of the generators are peripheral.

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